arXiv:math/9810084v1 [math.AG] 13 Oct 1998

M. P. APPELL'S FUNCTION AND VECTOR BUNDLES OF RANK 2 ON ELLIPTIC CURVES

A. Polishchuk[1]

Department of Mathematics
Harvard University, Cambridge, MA 02138

The purpose of this work is to establish a connection between the function

$$\kappa(y,x)=\kappa(y,x,\tau)=\sum_{n\in\mathbb{Z}}\frac{\exp(\pi i\tau n^2+2\pi inx)}{\exp(2\pi in\tau)-\exp(2\pi iy)},$$

where $y,x,\tau\in\mathbb{C}$, $\operatorname{Im}(\tau)>0$, $y\notin\mathbb{Z}+\mathbb{Z}\tau$, and vector bundles of rank 2 on elliptic curves. This function was introduced by M. P. Appell in [2] in order to decompose into simple elements the so called elliptic functions of the third kind (which correspond to meromorphic sections of line bundles on elliptic curves).[2]

As is well-known Riemann's theta function arises naturally when considering global sections of line bundles on elliptic curves. We claim that the function κ is connected in a similar way with rank-2 bundles. Namely, the difference equation for κ allows to interpret it as a global section of a rank-2 bundle of degree 1 on an elliptic curve. Pursuing this analogy we derive some interesting identities satisfied by κ similar to the addition formulas for θ. These identities (that were known to Appell) appear as A_∞-constraints in the Fukaya category of an elliptic curve (see [7]). Another identity for function κ turned out to be useful in proving some formulas from Ramanujan's Notebooks (cf. [4]). However, at the present moment there appears to be no general theory similar to that of theta-identities. The following pair of identities is derived easily from those known to Appell:

$$\theta(0)\kappa(\tau/2,1/2)+\theta(1/2)\kappa((\tau+1)/2,0)=\frac{1}{2}\theta(\tau/2)^3.$$

$$\theta(\tau/2)^3\kappa(1/2,\tau/2)=\theta(1/2)^3\kappa(\tau/2,1/2)+\theta(0)^3\kappa((\tau+1)/2,0).$$

[1]E-mail: apolish@math.harvard.edu

[2]The same function was introduced by M. Hermite essentially for the same problem, however, he didn't publish his results until the appearance of the first part of Appell's paper. In the second part of this paper Hermite's method is partially reproduced. Appell actually used the function which differs from κ by a theta-function. Our notation is closer to that of Halphen's book [5].

These identities relating three special values $\kappa(y,x)$ with y,x points of order 2 on elliptic curve $\mathbb{C}/\mathbb{Z}+\tau\mathbb{Z}$ deserve to be considered as analogues of the famous Jacobi identity relating the 4-th powers of values of θ at points of order 2. In fact, all the values $\kappa(x,y)$, where y and x are points of order 2 on elliptic curve, except these three can be expressed in terms of theta-function.

One may ask whether there is an analogue of modular property for the function κ similar to the functional equation for the theta function. Note that the latter equation reflects the fact that one can construct a line bundle on the universal elliptic curve (over the analytic moduli stack of elliptic curves with a non-trivial point of order 2) such that θ is its section. Similar, we can construct a vector bundle of rank 2 on the universal elliptic curve. However, it seems that there is no simple formula for the corresponding 1-cocycle of the modular group. So we propose to formulate the modular property of the function κ not in the form of the equation but in the form of divisibility property. More precisely, let us denote

$$\kappa_0(x,\tau)=\exp(\frac{3\pi i\tau}{4})\kappa((\tau+1)/2,x,\tau).$$

We prove that for every element $\gamma=\begin{pmatrix} a & b \\ c & d\end{pmatrix}$ of the group $\Gamma_{1,2}$ (this is the standard subgroup of $\mathrm{SL}_2(\mathbb{Z})$ which preserves the point $(\tau+1)/2$) the difference

$$\kappa_0(\frac{x}{c\tau+d},\frac{a\tau+b}{c\tau+d})-\zeta\cdot(c\tau+d)\cdot\exp(\pi i(\frac{1}{c\tau+d}-1)x)\kappa_0(x,\tau)$$

is divisible by $\theta(x,\tau)$ in the ring of holomorphic functions on $\mathfrak{H}\times\mathbb{C}$, where ζ is a root of unity of order 4 which we explicitly determine.

Although we concentrate on the case of bundles of degree 1, it seems that almost all facts concerning vector bundles of rank 2 on elliptic curves can be worked out explicitly in terms of the function κ (and elliptic functions). It seems that the case of higher rank bundles is described similarly by various functions derived from κ by taking derivatives and difference derivatives $(\kappa_{a_1}-\kappa_{a_2})/(a_1-a_2)$, etc.

In section 3 we show that some natural isomorphisms between rank-2 vector bundles can be written explicitly in terms of κ. As a byproduct we show how to find explicitly (in terms of κ) holomorphic functions ϕ_1 and ϕ_2 on $\mathbb{C}^*$ such that

$$\phi_1(z)\theta(z,q^2)-\phi_2(z)\theta(qz,q^2)=1$$

(one knows apriori that such functions exist since $\theta(z,q^2)$ and $\theta(qz,q^2)$ have no common zeroes).

We use both additive and multiplicative notations (i.e. sometimes we consider variables in $\mathbb{C}$ and sometimes in $\mathbb{C}^*$). The relation between (some) multiplicative and additive variables is the following: $z=\exp(2\pi ix)$, $q^{1/2}=\exp(\pi i\tau)$, $a=\exp(2\pi iy)$. By abuse of notation, for any function in multiplicative variables we will denote

the corresponding function in additive variables by the same letter (the only section where the additive notation is used is section 4 on modular property of κ). When working with a fixed elliptic curve $E_q = \mathbb{C}^*/q^{\mathbb{Z}}$ we often omit the variable q in the corresponding elliptic functions and the function κ.
Acknowledgment. I'd like to thank B. Berndt and G. Andrews for help with finding the references. This work is partially supported by NSF grant.

1. Bundles of rank 2 and degree 1

In this section we always use multiplicative variables.

1.1. Let $q \in \mathbb{C}^*$ be such that $|q| < 1$, $E = E_q = \mathbb{C}^*/q^{\mathbb{Z}}$ be the corresponding elliptic curve. For every invertible $r \times r$ matrix $A(z)$ of holomorphic functions on $\mathbb{C}^*$ we define the corresponding holomorphic vector bundle $V_r(A)$ on E_q as follows:

$$V_r(A) = \mathbb{C}^* \times \mathbb{C}^r/(z, v) \sim (qz, A(z)v).$$

Holomorphic sections of $V_r(A)$ correspond to r-tuple of functions $v(z)$ satisfying the equation

$$v(qz) = A(z)v(z).$$

The bundles $V_r(A)$ and $V_r(A')$ are isomorphic if and only if there exists an invertible $r \times r$ matrix $B(z)$ such that $A'(z) = B(qz)A(z)B(z)^{-1}$. The construction $A \mapsto V(A)$ is compatible with tensor products, in particular, if ϕ is an invertible holomorphic function on $\mathbb{C}^*$ then

$$V_r(\phi A) \simeq V_1(\phi) \otimes V_r(A).$$

1.2. We denote by L the line bundle $V_1(q^{-1/2}z^{-1})$. The classical theta function

$$\theta(z) = \theta(z, q) = \sum_{n \in \mathbb{Z}} q^{n^2/2} z^n$$

is a section of L (as function in z). We also denote by P_a the line bundle $V_1(a)$, where $a \in \mathbb{C}^*$ is considered as a constant function on $\mathbb{C}^*$. Then we have a canonical isomorphism

$$P_a \otimes t_a^* L \simeq L. \tag{1.2.1}$$

In particular, this implies that for every a one has

$$t_a^* L \otimes t_{a^{-1}}^* L \simeq L^2. \tag{1.2.2}$$

The classical addition formula for theta-function which we remind below gives a more concrete form of this isomorphism. The basis of global sections of L^2 consists of functions ϑ_0 and ϑ_1 defined as follows:

$$\vartheta_0(z) = \vartheta_0(z, q) = \theta(z^2, q^2) = \sum_{n \in \mathbb{Z}} q^{n^2} z^{2n},$$

$$\vartheta_1(z) = \vartheta_1(z,q) = \theta \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} (z^2, q^2) = \sum_{n\in\mathbb{Z}} q^{(n+1/2)^2} z^{2n+1}.$$

Now the addition formula corresponding to (1.2.2) is

$$\theta(za)\theta(za^{-1}) = \vartheta_0(a)\vartheta_0(z) + \vartheta_1(a)\vartheta_1(z). \tag{1.2.3}$$

1.3. Consider the rank-2 bundle

$$F_a = V_2 \begin{pmatrix} a & 1 \\ 0 & q^{-1/2}z^{-1} \end{pmatrix}.$$

This bundle is a unique non-trivial extension of L by P_a. Assume that $a \notin q^{\mathbb{Z}}$. Then $H^0(E, P_a) = H^1(E, P_a) = 0$, hence the natural projection $H^0(E, F_a) \to H^0(E, L)$ is an isomorphism. This means that there exists a unique global section of F_a projecting to $\theta(z,q)$. This section has form

$$\begin{bmatrix} \kappa_a(z) \\ \theta(z) \end{bmatrix}$$

where $\kappa_a(z) = \kappa(a,z) = \kappa(a,z,q)$ is the unique holomorphic in z function satisfying the equation

$$\kappa(a, qz, q) = a\kappa(a,z,q) + \theta(z,q). \tag{1.3.1}$$

This function can be written as the following series

$$\kappa(a,z,q) = \sum_{n\in\mathbb{Z}} \frac{q^{n^2/2}}{q^n - a} z^n$$

converging for all z, since $|q| < 1$. Using the defining equation (1.3.1) one can easily establish the following identities:

$$\kappa(a,z) = -a^{-1}\kappa(a^{-1}, qz^{-1}), \tag{1.3.2}$$

$$\kappa(qa,z) = q^{-1/2}z\kappa(a,qz) = q^{-1/2}az\kappa(a,z) + q^{-1/2}z\theta(z) \tag{1.3.3}$$

The latter one allows to consider the pair (κ, θ) as a meromorphic section of certain rank-2 bundle on $E \times E$. It is also convenient to consider the function

$$\bar{\kappa}(a,z) = \theta(-q^{-1/2}a)\kappa(a,z)$$

which is holomorphic in a and z. Comparing the quasi-periodicity properties of this function in a and z one immediately arrives to the following symmetry identity:

$$a\bar{\kappa}(a,z) = -q^{1/2}z\bar{\kappa}(-q^{1/2}z, -q^{-1/2}a). \tag{1.3.4}$$

Note that this identity appears as the main theorem of [4] where it is applied to derive some of Ramanujan's formulas. However, it can be found already in Appell's paper [2] (t.III, p.20) where it is shown to be equivalent to certain identity of Hermite.

Finally, the function $\bar{\kappa}(az, z^{-1})$ for fixed a is a section of $P_{-q^{-1/2}a^{-1}} \otimes L^2$. If we choose a square root $a^{1/2}$ we can write this as $t^*_{iq^{1/4}a^{1/2}}L^2$. Hence, we can express $\bar{\kappa}(az, z^{-1})$ as a linear combination of the corresponding shifts of ϑ_0 and ϑ_1. Using the vanishing $\vartheta_0(iq^{1/2}) = \vartheta_1(i) = 0$ we find explicitly the coefficients and get the following identity:

(1.3.5)

$$\bar{\kappa}(az, z^{-1}) = \frac{\vartheta_0(iq^{1/4}a^{1/2}z)}{\vartheta_0(i)}\bar{\kappa}(q^{-1/4}a^{1/2}, q^{1/4}a^{1/2}) + \frac{\vartheta_1(iq^{1/4}a^{1/2}z)}{\vartheta_1(iq^{1/2})}\bar{\kappa}(q^{1/4}a^{1/2}, q^{-1/4}a^{1/2})$$

1.4. The analogue of the property (1.2.1) for rank-2 bundles F_a is the following canonical isomorphism:

$$t_b^* F_a \simeq P_{b^{-1}} \otimes F_{ab}. \tag{1.4.1}$$

Indeed, the left hand side is

$$V\begin{pmatrix} a & 1 \\ 0 & b^{-1}q^{-1/2}z^{-1} \end{pmatrix}$$

while the right hand side is

$$V\begin{pmatrix} a & b^{-1} \\ 0 & b^{-1}q^{-1/2}z^{-1} \end{pmatrix}$$

Now the above isomorphism follows from the equality

$$\begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} a & 1 \\ 0 & b^{-1}q^{-1/2}z^{-1} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & b^{-1} \end{pmatrix} = \begin{pmatrix} a & b^{-1} \\ 0 & b^{-1}q^{-1/2}z^{-1} \end{pmatrix}.$$

Using (1.2.1) and (1.4.1) we obtain an isomorphism

$$L \otimes t_b^* F_a \widetilde{\rightarrow} t_b^* L \otimes F_{ab}. \tag{1.4.2}$$

Moreover, this isomorphism is compatible with morphisms of both bundles to $t_b^*L \otimes L$. Now the section

$$\begin{bmatrix} \theta \cdot t_b^* \kappa_a \\ \theta \cdot t_b^* \theta \end{bmatrix}$$

of $L \otimes t_b^* F_a$ is mapped by (1.4.2) to the section

$$\begin{bmatrix} \theta \cdot t_b^* \kappa_a \\ b^{-1}\theta \cdot t_b^* \theta \end{bmatrix}$$

of $t_b^* L \otimes F_{ab}$. On the other hand, the latter bundle has the global section

$$\begin{bmatrix} \kappa_{ab} \cdot t_b^* \theta \\ \theta \cdot t_b^* \theta \end{bmatrix}.$$

It follows that the difference

$$\kappa_{ab} \cdot t_b^*\theta - b^{-1}\theta \cdot t_b^*\kappa_a$$

is a constant multiple of $t_{a^{-1}}^*\theta$. Considering the value of this difference at $z = -q^{1/2}$ and using vanishing $\theta(-q^{1/2}) = 0$ we get the following identity

$$\theta(bz)\kappa(ab, z) - b^{-1}\theta(z)\kappa(a, bz) = \frac{\theta(-q^{1/2}b)\theta(a^{-1}z)}{\theta(-q^{-1/2}a)}\kappa(ab, -q^{1/2}). \tag{1.4.3}$$

This identity appears in a slightly different form in Halphen's book [5] (p.481, formula (45) and the next one). Note that $\bar{\kappa}(a, -q^{1/2}) = a\bar{\kappa}(a, -q^{-1/2})$ and (1.3.4) gives

$$a\bar{\kappa}(a, -q^{-1/2}) = \bar{\kappa}(1, -q^{-1/2}a).$$

Now using the explicit series for $\kappa(a, z)$ we immediately see that

$$\bar{\kappa}(1, z) = \lim_{a \to 1} \frac{\theta(-q^{-1/2}a)}{1-a} = q^{-1/2}\frac{d\theta}{da}(-q^{-1/2}) = \frac{1}{2}\theta(1)\theta(-1)\theta(q^{1/2})$$

due to Jacobi's derivative formula. It follows that

$$\kappa(a, -q^{1/2}) = \frac{\theta(1)\theta(-1)\theta(q^{1/2})}{2\theta(-q^{-1/2}a)}. \tag{1.4.4}$$

Substituting this into (1.4.3) we get

$$\theta(bz)\kappa(ab, z) - b^{-1}\theta(z)\kappa(a, bz) = \frac{\theta(1)\theta(-1)\theta(q^{1/2})\theta(-q^{1/2}b)\theta(a^{-1}z)}{2\theta(-q^{-1/2}a)\theta(-q^{-1/2}ab)}. \tag{1.4.5}$$

This identity can be considered as a rank-2 analogue of the addition formula (1.2.3). It allows to express κ in terms of $\kappa(q^{1/2}, z)$ and theta-functions:

(1.4.6)

$$\kappa(a, z) = q^{1/2}a^{-1}\frac{\theta(z)}{\theta(q^{-1/2}az)}\kappa(q^{1/2}, q^{-1/2}az) + \frac{\theta(1)\theta(q^{1/2})\theta(-a)\theta(q^{-1/2}z)}{2\theta(-q^{-1/2}a)\theta(q^{-1/2}az)}.$$

1.5. We have a canonical morphism

$$\mu_{a,b} : H^0(E, t_{b^{-1}}^*L) \otimes H^0(E, t_b^*F_a) \to H^0(E, t_{b^{-1}}^*L \otimes t_b^*F_a) \simeq H^0(E, L \otimes F_{ab}).$$

Assume that a and ab are not integer powers of q. Then this map is given by

$$\mu_{a,b}(t_{b^{-1}}^*\theta \otimes \begin{bmatrix} t_b^*\kappa_a \\ t_b^*\theta \end{bmatrix}) = \begin{bmatrix} t_{b^{-1}}^*\theta \cdot t_b^*\kappa_a \\ bt_{b^{-1}}^*\theta \cdot t_b^*\theta \end{bmatrix}.$$

Using the maps $\mu_{a,b}$ one can easily construct a basis in $H^0(E, L \otimes F_a)$. Indeed, assume that $a \in \mathbb{C}^*$ is such that $a \notin q^{\mathbb{Z}}$ and $-a \notin q^{\mathbb{Z}}$. Then we claim that $H^0(E, L \otimes F_a)$ is

generated by the images of $\mu_{a,1}$, $\mu_{-a,-1}$ and the subspace $H^0(E, L\otimes P_a)$. This follows immediately from the exact sequence

$$0 \to H^0(E, L\otimes P_a) \to H^0(E, L\otimes F_a) \to H^0(E, L^2) \to 0. \tag{1.5.1}$$

and the fact that θ^2 and $t_{-1}^*\theta^2$ generate $H^0(E, L^2)$. It follows that the following vectors form a basis of $H^0(E, L\otimes F_a)$:

$$v_0(a) = \begin{bmatrix}\theta(a^{-1}z)\\ 0\end{bmatrix}, v_1(a) = \begin{bmatrix}\theta(z)\cdot\kappa(a,z)\\ \theta(z)^2\end{bmatrix}, v_{-1}(a) = \begin{bmatrix}\theta(-z)\cdot\kappa(-a,-z)\\ -\theta(-z)^2\end{bmatrix}.$$

One can write the map $\mu_{a,b}$ expicitly in terms of this basis using (1.4.6). Namely, one has

$$b^{-1}\cdot\mu_{a,b}(t_{b^{-1}}^*\theta\otimes\begin{bmatrix}t_b^*\kappa_a\\ t_b^*\theta\end{bmatrix}) = \lambda_b\cdot v_1(ab) - \lambda_{-b}\cdot v_{-1}(ab) + (\nu_{a,b}-\nu_{a,-b})\cdot v_0(ab)$$

where

$$\lambda_b = \frac{\theta(q^{1/2}b^{-1})\theta(q^{1/2}b)}{\theta(q^{1/2})^2},$$

$$\nu_{a,b} = \frac{\theta(1)\theta(q^{1/2}b)\theta(-q^{1/2}b)\theta(-q^{1/2}b^{-1})\theta(b)\theta(ab)}{2\theta(q^{1/2})\theta(-q^{1/2}a^{-1})\theta(q^{-1/2}ab)\theta(-ab^2)}.$$

2. Various formulas

2.1. Let us consider special values of $\bar\kappa(a,z)$ for $a, z\in\{\pm1, \pm q^{1/2}\}$. It is easy to see that all of them except for $\bar\kappa(q^{1/2},-1)$, $\bar\kappa(-q^{1/2},1)$ and $\bar\kappa(-1,q^{1/2})$ can be expressed in terms of values of the theta function. We have already seen how to compute $\kappa(a,-q^{1/2})$. For example,

$$\kappa(-q^{1/2},-q^{1/2}) = \frac{1}{2}\theta(-1)\theta(q^{1/2}), \tag{2.1.1}$$

$$\kappa(-1,-q^{1/2}) = \frac{1}{2}\theta(1)\theta(-1). \tag{2.1.2}$$

The latter identity appears as entry 18 in [3], p.152. Also using (1.3.2) and (1.3.4) one can easily deduce that

$$\kappa(-1,\pm1) = \frac{1}{2}\theta(\pm1), \tag{2.1.3}$$

$$\kappa(\pm q^{1/2}, q^{1/2}) = \frac{1}{2}\theta(q^{1/2}), \tag{2.1.4}$$

while $\kappa(q^{1/2},1)=\kappa(-q^{1/2},-1)=0$ as one can see immediately from the formulas

$$\kappa(q^{1/2},z)=\sum_{n\geq 0}\frac{q^{n^2/2+n}}{1-q^{n+1/2}}(z^{-n}-z^{n+1}),$$

$$\kappa(-q^{1/2},z)=\sum_{n\geq 0}\frac{q^{n^2/2+n}}{1+q^{n+1/2}}(z^{-n}+z^{n+1}).$$

Note that three exceptional values correspond to pairs (a,z) which are stable under involution $(a,z)\mapsto(-q^{1/2}z,-q^{-1/2}a)$ (modulo $q^{\mathbb{Z}}$).

2.2. Using the above information about the special values of κ one can derive from (1.4.6) the following identities

$$2\kappa(q^{1/2}b^{-1},q^{1/2}b)=\theta(q^{-1/2}b)+\frac{\theta(1)\theta(b)}{\theta(-b)}\theta(-q^{1/2}b). \tag{2.2.1}$$

$$\kappa(q^{1/2}b^{-1},b)=\frac{\theta(q^{1/2})\theta(q^{-1/2}b)\theta(-q^{-1/2}b)}{2\theta(-b)}=\frac{(q)_\infty^2(-q)_\infty^2(-b)_\infty(-qb^{-1})_\infty(b)_\infty(qb^{-1})_\infty}{(q^{1/2}b)_\infty(q^{1/2}b^{-1})_\infty}. \tag{2.2.2}$$

$$\theta(-z)\kappa(a,z)+\theta(z)\kappa(-a,-z)=\frac{\theta(q^{1/2})^2\theta(1)\theta(-1)\theta(-a^{-1}z)}{2\theta(q^{1/2}a^{-1})\theta(-q^{1/2}a^{-1})}. \tag{2.2.3}$$

Here is another identity which follows from (1.4.6):

$$\theta(q^{1/2})^2\theta(-b)\kappa(q^{1/2}b^{-1},-b)=\theta(q^{1/2}b^{-1})^2\theta(-1)\kappa(q^{1/2},-1)+\theta(-q^{1/2}b^{-1})^2\theta(1)\kappa(-q^{1/2},1) \tag{2.2.4}$$

2.3. The special values $\kappa(q^{1/2},-1)$, $\kappa(-q^{1/2},1)$ and $\kappa(-1,q^{1/2})$, which we were not able to express in terms of theta-functions, satisfy the following two identities:

$$\theta(1)\kappa(q^{1/2},-1)+\theta(-1)\kappa(-q^{1/2},1)=\frac{1}{2}\theta(q^{1/2})^3. \tag{2.3.1}$$

$$\theta(q^{1/2})^3\kappa(-1,q^{1/2})=\theta(-1)^3\kappa(q^{1/2},-1)+\theta(1)^3\kappa(-q^{1/2},1). \tag{2.3.2}$$

The first formula is obtained by substituting $b=z=-1$, $a=-q^{1/2}$ into (1.4.5), while (2.3.2) is the specialization of (2.2.4) for $b=-q^{1/2}$. We can rewrite (2.3.1) as

follows

$$(\sum_{k\in\mathbb{Z}} q^{k^2/2}) \left(\sum_{n\geq 0}(-1)^n\frac{q^{n^2/2+n}}{1-q^{n+1/2}}\right) + (\sum_{k\in\mathbb{Z}}(-1)^k q^{k^2/2}) \left(\sum_{n\geq 0}\frac{q^{n^2+n}}{1+q^{n+1/2}}\right) =$$
$$2\left(\sum_{n\geq 0} q^{(n^2+n)/2}\right)^3.$$

The LHS can be further rewritten as

$$\sum_{n\geq 0, k\in n+2\mathbb{Z}}(-1)^n q^{k^2/2+n^2/2+n}(\frac{1}{1-q^{n+1/2}}+\frac{1}{1+q^{n+1/2}})+$$
$$\sum_{n\geq 0, k\in n+1+2\mathbb{Z}}(-1)^n q^{k^2/2+n^2/2+n}(\frac{1}{1-q^{n+1/2}}-\frac{1}{1+q^{n+1/2}}) =$$
$$2\sum_{n\geq 0, l\in\mathbb{Z}}(-1)^n\frac{q^{(n+2l)^2/2+n^2/2+n}}{1-q^{2n+1}} + 2\sum_{n\geq 0, l\in\mathbb{Z}}(-1)^n\frac{q^{n-2l+1)^2/2+n^2/2+2n+1/2}}{1-q^{2n+1}}.$$

Simplifying we get the identity

$$\left(\sum_{n\geq 0} q^{(n^2+n)/2}\right)^3 = \sum_{n\geq 0, l\in\mathbb{Z}}(-1)^n\frac{q^{n^2+n-2nl+2l^2}(1+q^{2l+1})}{1-q^{2n+1}}$$

Note that G. Andrews in [1] has obtained another formula for $(\sum q^{(n^2+n)/2})^3$, namely

$$\left(\sum_{n\geq 0} q^{(n^2+n)/2}\right)^3 = \sum_{n\geq 0, 2n\geq j\geq 0}\frac{q^{2n^2+2n-j(j+1)/2}(1+q^{2n+1})}{1-q^{2n+1}}$$

where all the coefficients in the RHS are positive (which in particular proves that every number is a sum of three triangular numbers). It would be interesting to see directly why the right hand sides in two formulas are equal.

3. Isomorphisms between bundles

In this section we show that some isomorphisms between rank-2 bundles can be written explicitly using the function κ.

3.1. The first isomorphism we are interested in is an isomorphism of F_a (see section 1) and the unique non-trivial extension of $\det F_a \simeq P_a\otimes L$ by $\mathcal{O}$. The latter extension is realized by the rank-2 bundle

$$F'_a = V_2\begin{pmatrix}1 & 1\\ 0 & q^{-1/2}az^{-1}\end{pmatrix}.$$

To give an isomorphism $F'_a \simeq F_a$ we have to find a 2×2 matrix of holomorphic functions $B(z)$ with the property

$$\begin{pmatrix} a & 1 \\ 0 & q^{-1/2}z^{-1} \end{pmatrix} = B(qz) \begin{pmatrix} 1 & 1 \\ 0 & q^{-1/2}az^{-1} \end{pmatrix} B(z)^{-1}. \tag{3.1.1}$$

Since B should send global sections of F'_a to global sections of F_a we can choose B in the form

$$B = \begin{pmatrix} \kappa_a(z) & f(z) \\ \theta(z) & g(z) \end{pmatrix}.$$

Now it is easy to see that (3.1.1) implies

$$g(qz) = a^{-1}g(z) - a^{-1}\theta(z),$$

hence, $g(z) = -a^{-1}\kappa_{a^{-1}}(z)$. To find $f(z)$ we note that the determinant of B should be q-periodic, hence, a constant function. Therefore, we have

$$f(z)\theta(z) = c - a^{-1}\kappa_a(z)\kappa_{a^{-1}}(z)$$

for some constant c. Substituting $z = -q^{1/2}$ we find using (1.4.4) that

$$c = c_a = a^{-1}\kappa_a(-q^{1/2})\kappa_{a^{-1}}(-q^{1/2}) = \frac{\theta(1)^2\theta(-1)^2\theta(q^{1/2})^2}{4a\theta(-q^{-1/2}a)\theta(-q^{1/2}a)},$$

and the matrix B is equal to

$$\begin{pmatrix} \kappa_a(z) & \frac{c_a - a^{-1}\kappa_a(z)\kappa_{a^{-1}}(z)}{\theta(z)} \\ \theta(z) & -a^{-1}\kappa_{a^{-1}}(z) \end{pmatrix}.$$

3.2. Now we want to compare bundles $F_a \simeq F'_a$ with push-forwards of line bundles of degree 1 on $E_{q^2} = \mathbb{C}^*/q^{2\mathbb{Z}}$ under the natural isogeny $\pi : E_{q^2} \to E_q$. It is sufficient to do this for F'_1 since F'_a is obtained from F'_1 by translation. We claim that there is an isomorphism

$$F'_1 \simeq \pi_* L'$$

where L' is a line bundle on E_{q^2} with multiplicator $-q^{-1/2}z^{-1}$, i.e.

$$L' = V_1^{q^2}(-q^{-1/2}z^{-1}),$$

where $V_1^{q^2}$ is the construction of section 1 with q replaced by q^2. Indeed, the unique section of L' (which is given by $\theta(-q^{-1/2}z, q^2)$) induces the global non-vanishing section of $\pi_* L'$, so $\pi_* L'$ fits into exact sequence

$$0 \to \mathcal{O} \to \pi_* L' \to L \to 0.$$

On the other hand, it is easy to check that $\pi_* L'$ is a simple bundle, hence the above sequence doesn't split and $\pi_* L'$ is isomorphic to F_1'. To write this isomorphism explicitly note that

$$\pi_* L' = V_2 \begin{pmatrix} 0 & -q^{-1/2}z^{-1} \\ 1 & 0 \end{pmatrix}.$$

Thus, we are looking for a matrix $C(z)$ such that

$$\begin{pmatrix} 1 & 1 \\ 0 & q^{-1/2}z^{-1} \end{pmatrix} = C(qz) \begin{pmatrix} 0 & -q^{-1/2}z^{-1} \\ 1 & 0 \end{pmatrix} C(z)^{-1}. \tag{3.2.1}$$

Let $C(z) = \begin{pmatrix} c_{11}(z) & c_{12}(z) \\ c_{21}(z) & c_{22}(z) \end{pmatrix}$. Then from (3.2.1) we deduce that the vector

$$\begin{bmatrix} c_{22}(z) \\ -c_{21}(z) \end{bmatrix}$$

is a global section of $\pi_* L'$. Hence, we can choose C with $c_{21}(z) = \theta(-q^{-1/2}z, q^2)$ and $c_{22}(z) = -c_{21}(qz)$. As before (3.2.1) implies that $\det(C)$ is a (non-zero) constant, i.e.

$$c_{11}(z)c_{21}(qz) + c_{12}(z)c_{21}(z) = c \tag{3.2.2}$$

for some constant c. Another consequence of (3.2.1) is that $c_{11}(z) = c_{12}(qz) - c_{21}(z)$. Substituting this in (3.2.2) we get

$$c_{12}(qz)c_{21}(qz) = -c_{12}(z)c_{21}(z) + c + c_{21}(z)c_{21}(qz).$$

In other words, if we denote $\phi(z) = c_{12}(z)c_{21}(z) - c/2$ then we have

$$\phi(qz) = -\phi(z) + c_{21}(z)c_{21}(qz).$$

Note that

$$c_{21}(z)c_{21}(qz) = \lambda \cdot \theta(-z, q)$$

where

$$\lambda = \frac{\theta(1, q^2)\theta(q, q^2)}{\theta(q^{1/2}, q)}.$$

It follows that $\phi(z) = \lambda \cdot \kappa_{-1}(-z, q)$, i.e.

$$c_{12}(z)c_{21}(z) = \frac{c}{2} + \lambda \cdot \kappa_{-1}(-z, q).$$

Substituting $z = q^{-1/2}$ (zero of c_{21}) we obtain

$$\frac{c}{2} = -\lambda \cdot \kappa_{-1}(-q^{-1/2}, q) = \lambda \cdot \kappa(-1, -q^{1/2}, q) = \lambda \cdot \frac{\theta(1,q)\theta(-1,q)}{2}$$

(in the last equality we used (2.1.2)). Similarly, we find

$$c_{11}(z)c_{21}(qz) = \frac{c}{2} - \lambda \cdot \kappa_{-1}(-z, q).$$

So finally the matrix C is equal to

$$\begin{pmatrix} \lambda\cdot\frac{\theta(1,q)\theta(-1,q)/2-\kappa_{-1}(-z,q)}{\theta(-q^{1/2}z,q^2)} & \lambda\cdot\frac{\theta(1,q)\theta(-1,q)/2+\kappa_{-1}(-z,q)}{\theta(-q^{-1/2}z,q^2)} \\ \theta(-q^{-1/2}z,q^2) & -\theta(-q^{1/2}z,q^2) \end{pmatrix}.$$

Note that since $\det C$ is a non-zero constant as a byproduct we can find explicitly holomorphic functions ϕ_1, ϕ_2 on $\mathbb{C}^*$ with the property

$$\phi_1(z)\theta(z,q^2)-\phi_2(z)\theta(qz,q^2)=1.$$

4. Modular property

Throughout this section we use additive notation.

4.1. Let us recall the modular interpretation of the functional equation for theta-function. For this we have to consider the stack $\mathcal{A}_1^+$ of elliptic curves (over $\mathbb{C}$) equipped with a non-trivial point of order 2. It is well-known that $\mathcal{A}_1^+$ is the quotient of the upper-half plane $\mathfrak{H}$ by the action of the subgroup $\Gamma_{1,2}\subset \mathrm{SL}_2(\mathbb{Z})$ consisting of matrices $\gamma=\begin{pmatrix} a & b\\ c & d\end{pmatrix}$ with $\det\gamma=1$ and $ac\equiv bd\equiv 0 \mod (2)$. The universal elliptic curve $\mathcal{E}\to\mathcal{A}_1^+$ is the quotient of $\mathbb{C}\times\mathfrak{H}$ by the natural action of the semi-direct product of $\mathbb{Z}^2$ with $\Gamma_{1,2}$, where the action of $\mathbb{Z}^2$ is given by

$$(x,\tau)\mapsto(x+m+n\tau,\tau)$$

while the above matrix in $\Gamma_{1,2}$ acts by

$$\gamma:(x,\tau)\mapsto(\frac{x}{c\tau+d},\gamma(\tau)=\frac{a\tau+b}{c\tau+d}).$$

Let $\pi:\mathbb{C}\times\mathfrak{H}\to\mathcal{E}$ be the natural projection. Below we will deal with holomorphic vector bundles V on $\mathcal{E}$ (of ranks 1 and 2) such that π^*V is trivial. Fixing such a trivialization gives a 1-cocycle of $G=\mathbb{Z}^2\rtimes\Gamma_{1,2}$ with values in the group of holomorphic functions $\mathbb{C}\times\mathfrak{H}\to\mathrm{GL}_r(\mathbb{C})$, where the action of G on $\mathbb{C}\times\mathfrak{H}$ is as above, $r=\mathrm{rk}\,V$. Conversely, given such a 1-cocycle $c(g)(x,\tau)$, where $g\in G$ we can construct a vector bundle on $\mathcal{E}$ as the quotient of $\mathbb{C}\times\mathfrak{H}\times\mathbb{C}^r$ by the action of G which sends $((x,\tau),v)$ to $(g(x,\tau),c(g)(x,\tau)v)$. Thus, the set of isomorphism classes of holomorphic bundles on $\mathcal{E}$ with trivial pull-back to $\mathbb{C}\times\mathfrak{H}$ can be identified with $H^1(G,\mathrm{GL}_r(\mathcal{O}(\mathbb{C}\times\mathfrak{H})))$. Since G is a semi-direct product of $\mathbb{Z}^2$ and $\Gamma_{1,2}$, any 1-cocycle of G is uniquely determined by its restrictions to $\mathbb{Z}^2$ and $\Gamma_{1,2}$ (these restricted cocycles should be compatible via the action of $\Gamma_{1,2}$ on $\mathbb{Z}^2$). For example, $\theta(x,\tau)$ is a global section of the line bundle $\mathcal{L}$ on $\mathcal{E}$ given by a cocycle whose restriction to $\mathbb{Z}^2$ is

$$(m,n)\mapsto\exp(-\pi i n^2\tau-2\pi i n x),$$

and the restriction to $\Gamma_{1,2}$ is

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \zeta(\gamma) \cdot (c\tau + d)^{1/2} \exp(\pi i \frac{cx^2}{c\tau + d}) \tag{4.1.1}$$

where $\zeta(\gamma)$ is the 8-th root of unity appearing in the functional equation for θ (the latter equation expresses the fact that θ is a section of $\mathcal{L}$). Note that the cocycle (4.1.1) factors into a product of two cocycles of $\Gamma_{1,2}$, the first is with values in $\mathcal{O}^*(\mathfrak{H}) \subset \mathcal{O}^*(\mathbb{C} \times \mathfrak{H})$:

$$\gamma \mapsto c_\theta(\gamma) = \zeta(\gamma) \cdot (c\tau + d)^{1/2},$$

and the second is given by the exponential factor. Actually, c_θ is a coboundary since $\theta(0, \tau)$ is an invertible function on $\mathfrak{H}$ and

$$c_\theta(\gamma) = \frac{\theta(0, \gamma(\tau))}{\theta(0, \tau)}.$$

We can't factor c_θ further due to indeterminacy in a choice of a square root of $(c\tau + d)$. However, c_θ^2 does factor as the product of a character $\gamma \mapsto \zeta(\gamma)^2$ of $\Gamma_{1,2}$ with values in roots of unity of order 4, and a cocycle $\gamma \mapsto (c\tau + d)$ which corresponds to the line bundle ω^{-1} on $\mathcal{A}_1^+$ (where ω is the restriction of the relative canonical bundle $\omega_{\mathcal{E}/\mathcal{A}_1^+}$ to the zero section). In particular, since c_θ^2 is a coboundary we obtain that ω is isomorphic to the line bundle on $\mathcal{A}_1^+$ associated with the character $\zeta(\gamma)^2$. The explicit formula for $\zeta(\gamma)$ (see e.g. [6]) implies that

$$\zeta(\gamma)^2 = \begin{cases} (-1)^{(d-1)/2}, & d \text{ odd}, \\ \exp(-\frac{\pi i c}{2}), & c \text{ odd} \end{cases}$$

4.2. Now we are going to consider a rank-2 bundle on $\mathcal{E}$ which is an extension of $\mathcal{L}$ by certain line bundle whose restriction to every elliptic curve is of 2-torsion. More presicely, let L be a restriction of $\mathcal{L}$ to a particular elliptic curve E. Then we have $L \simeq \mathcal{O}(\eta)$ where $\eta \in E$ is the point of order 2 corresponding to $(\tau + 1)/2$ (this isomorphism is induced by the theta-function). Now let $P_\eta = \mathcal{O}(\eta - e) \otimes \omega_E^{-1}$ be the 2-torsion line bundle corresponding to η, trivialized at zero $e \in E$. Then we have

$$H^1(L^{-1} \otimes P_\eta) \simeq H^1(\mathcal{O}(-e) \otimes \omega_E^{-1}) \simeq H^0(\omega_E^2)^*.$$

It follows that there is a canonical non-splitting extension

$$0 \to P_\eta \otimes \omega_E^2 \to V \to L \to 0.$$

In other words, there should exist a universal extension on $\mathcal{E}$

$$0 \to \mathcal{P}_\eta \otimes \omega^2 \to \mathcal{V} \to \mathcal{L} \to 0$$

where $\mathcal{P}_\eta$ is defined in the same way as P_η with η being the divisor $x = (\tau + 1)/2 \mod (\mathbb{Z} + \mathbb{Z}\tau)$, e being the zero section, and ω_E replaced by ω. The pull-back of $\mathcal{P}_\eta$ to $\mathbb{C} \times \mathfrak{H}$ is trivial and the corresponding 1-cocycle of G is given by

$$(m, n) \mapsto \exp(\pi i n(\tau + 1)),$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \exp(\pi i(\frac{1}{c\tau + d} - 1)x).$$

Below we construct a bundle $\mathcal{V}$ on $\mathcal{E}$ fitting into above exact sequence together with a trivialization of $\pi^*\mathcal{V}$ such that the pair (κ, θ) defines a global section of $\mathcal{V}$. Note that one has a similar rank-2 bundle on the stack of elliptic curves (without a choice of a point of order 2), which is an extension of $\mathcal{O}(e)$ by ω. However, we will stick to the case of $\mathcal{A}_1^+$ in order to stress the analogy with the usual functional equation for θ (which takes place on this stack).

4.3. Recall that for an elliptic curve $E = E^\tau = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$ we have defined in section 1 the rank-2 bundle on E^τ

$$V^\tau = F_{\exp(\pi i(\tau+1))} = V_2 \begin{pmatrix} \exp(\pi i(\tau + 1)) & 1 \\ 0 & \exp(-\pi i\tau - 2\pi i x) \end{pmatrix}$$

which is an extension of $L = L^\tau$ by $P_\eta = P_\eta^\tau$. Now for an element

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_{1,2}$$

we have the corresponding isomorphism

$$f_\gamma : E^\tau \to E^{\gamma(\tau)} : x \mapsto \frac{x}{c\tau + d}$$

induced by the action of $\Gamma_{1,2}$ on $\mathbb{C} \times \mathfrak{H}$. The 1-cocycle of G corresponding to $\mathcal{L}$ (resp. $\mathcal{P}_\eta$) evaluated at γ induces an isomorphism $f_\gamma^* L^{\gamma(\tau)} \simeq L^\tau$ (resp. $f_\gamma^* P_\eta^{\gamma(\tau)} \simeq P_\eta^\tau$). Using these isomorphisms we consider the pull-back by f_γ of the extension

$$0 \to P_\eta^{\gamma(\tau)} \to V^{\gamma(\tau)} \to L^{\gamma(\tau)} \to 0$$

as an extension of L^τ by P_η^τ. The class of the latter extension differs from the class of the extension given by V^τ by a non-zero constant. Hence, there exists a unique isomorphism

$$V^\tau \widetilde{\to} f_\gamma^* V^{\gamma(\tau)}$$

which induces the identity map on L^τ and a a non-zero constant multiple of the identity on P_η^τ.

Equivalently, one can say that there exists a unique 1-cocycle of G with values in $\mathrm{GL}_2(\mathcal{O}(\mathbb{C}\times\mathfrak{H}))$ such that its restriction to $\mathbb{Z}^2$ is

$$(m,n)\mapsto\begin{pmatrix}\exp(\pi in(\tau+1)) & 1\\ 0 & \exp(-\pi in^2\tau-2\pi inx)\end{pmatrix}\tag{4.3.1}$$

and its restriction to $\Gamma_{1,2}$ has form

$$\gamma=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\mapsto\begin{pmatrix}k_\gamma(\tau)\cdot\exp(\pi i(\frac{1}{c\tau+d}-1)x) & \phi_\gamma(x,\tau)\\ 0 & c_\theta(\gamma)\cdot\exp(\pi i\frac{cx^2}{c\tau+d})\end{pmatrix}\tag{4.3.2}$$

for some holomorphic function $\phi_\gamma(x,\tau)$ and some invertible holomorphic function $k_\gamma(\tau)$. Moreover, it follows from definition of κ that

$$\begin{bmatrix}\kappa((\tau+1)/2,x,\tau)\\ \theta(x,\tau)\end{bmatrix}$$

is a section of the bundle $\mathcal{V}$ defined by the above cocycle. This allows us to find functions k_γ and ϕ_γ explicitly. Namely, for any $\gamma\in\Gamma_{1,2}$ we should have

(4.3.3)

$$\kappa((\gamma(\tau)+1)/2,\frac{x}{c\tau+d},\gamma(\tau))=k_\gamma(\tau)\cdot\exp(\pi i(\frac{1}{c\tau+d}-1)x)\cdot\kappa((\tau+1)/2,x,\tau)+\phi_\gamma(x,\tau)\theta(x,\tau).$$

To find k_γ let us substitute $x=(\tau+1)/2$ in this equation. Note that the equation (1.3.1) together with the vanishing $\theta((\tau+1)/2,\tau)=0$ implies that

$$\kappa((\tau+1)/2,x+m+n\tau,\tau)=\exp(\pi in(\tau+1))\kappa((\tau+1)/2,x,\tau).$$

Using this together with the formulas

$$\frac{1}{c\tau+d}=a-c\gamma(\tau),$$

$$\frac{\tau}{c\tau+d}=-b+d\gamma(\tau)$$

we get from (4.3.3)

$$\exp(\pi i\frac{d-c-1}{2}(\gamma(\tau)+1))\kappa((\gamma(\tau)+1)/2,(\gamma(\tau)+1)/2,\gamma(\tau))=$$
$$k_\gamma(\tau)\cdot\exp(\frac{\pi i}{2}(a-b-1+(d-c)\gamma(\tau)-\tau))\cdot\kappa((\tau+1)/2,(\tau+1)/2,\tau),$$

i.e.

$$k_\gamma=\exp(\frac{\pi i}{2}(d-a+b-c-\gamma(\tau)+\tau))\cdot\frac{\kappa((\gamma(\tau)+1)/2,(\gamma(\tau)+1)/2,\gamma(\tau))}{\kappa((\tau+1)/2,(\tau+1)/2,\tau)}.$$

Now using the formula (2.1.1) for $\kappa((\tau+1)/2,(\tau+1)/2,\tau)$ and the functional equation for the theta-function we obtain

$$k_\gamma = \exp(\frac{\pi i}{2}(d-a+b-c)+\frac{3\pi i}{4}(\tau-\gamma(\tau)))\cdot\zeta(\gamma)^2\cdot\chi(\gamma)\cdot(c\tau+d)$$

where χ is the character of $\Gamma_{1,2}$ with values in 4th roots of unity defined as follows:

$$\chi(\gamma)=\begin{cases}(-1)^{a/2}\exp(\frac{\pi i}{4}(ab+cd)), & a \text{ even},\\ (-1)^{c/2}\exp(\frac{\pi i}{4}(ab+cd)), & c \text{ even}\end{cases}$$

It is easy to see that $(-1)^{(d-a+b-c)/2}=\chi(\gamma)^2\zeta(\gamma)^4$, so we can write finally

$$k_\gamma = \exp(\frac{3\pi i}{4}(\tau-\gamma(\tau)))\cdot\zeta(\gamma)^{-2}\cdot\chi^{-1}(\gamma)\cdot(c\tau+d) \tag{4.3.4}$$

Now the equation (4.3.3) gives the explicit formula for ϕ_γ.

Summarizing we can say that the existence of the rank-2 bundle $\mathcal{V}$ on the universal elliptic curve $\mathcal{E}$ is equivalent to the following divisibility property of κ. Let us denote

$$\kappa_0(x,\tau)=\exp(\frac{3\pi i\tau}{4})\kappa((\tau+1)/2,x,\tau).$$

Then for every $\gamma\in\Gamma_{1,2}$ the difference

$$\kappa_0(\frac{x}{c\tau+d},\gamma(\tau))-\zeta(\gamma)^{-2}\cdot\chi^{-1}(\gamma)\cdot(c\tau+d)\cdot\exp(\pi i(\frac{1}{c\tau+d}-1)x)\cdot\kappa_0(x,\tau)$$

is divisible by $\theta(x,\tau)$.

References

1. G. E. Andrews, *The fifth and seventh order mock theta functions*. Trans. AMS 293 (1986), 113–134.
2. M. P. Appell, *Sur le fonctions doublement periodique de troisieme espece*, Annales scientifiques de l'École Normale Supérieure, 3e série, t.I, p.135, t.II, p.9, t.III, p.9 (1884–1886).
3. B. C. Berndt, *Ramanujan's Notebooks, Part IV*. Springer-Verlag, 1994.
4. R. Evans, *Generalized Lambert series*, in *Analytic number theory*, Proceedings of a conference in honor of Heini Halberstam, ed. B. Berndt et al., vol.1, 357–370. Birkhauser, 1996.
5. G.-H. Halphen, *Traité des fonctions elliptiques*, I. Paris, 1886.
6. D. Mumford, *Tata lectures on theta I*, Birkhäuser, 1983.
7. A. Polishchuk, *Massey and Fukaya products on elliptic curves*, preprint math.AG/9803017.